SUDDEN TRANSITIONS TO CHAOS IN A SEMICONDUCTOR LASER WITH OPTICAL DELAY

K. GREEN*

Department of Computer Science
Katholieke Universiteit Leuven
Celestijnenlaan 200A
3001 Heverlee, Belgium
E-mail: kirk.green@cs.kuleuven.ac.be

B. KRAUSKOPF†

Department of Engineering Mathematics
University of Bristol
Bristol BS8 1TR, UK
E-mail: b.krauskopf@bristol.ac.uk

Work is presented of how a new method to compute 1D unstable manifolds of saddle periodic orbits of delay equations can be used to identify transitions to chaos in a physical system that is subject to delayed feedback. Specifically, we study an interior crisis bifurcation and an intermittent transition in a semiconductor laser subject to phase-conjugate feedback.

Sudden transitions to chaos in dynamical systems are characterised by jumps in the size of the attracting solution, for example, from an attracting periodic solution to a much larger chaotic attractor. Generally called crisis bifurcations,[8] this phenomenon is due to a rearrangement of the stable and unstable manifolds of suitable saddle points. Advances in numerical tools allowing the computation of stable and unstable manifolds in systems described by maps[6] have led to greater insight into such bifurcations.[7]

Recently, we developed the first algorithm for computing one-dimensional unstable manifolds of a fixed point of a suitable Poincaré map in delay differential equations (DDEs).[4] Briefly, this method computes the

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manifold as a sequence of points in a suitable plane of intersection transverse to the flow. A linear approximation for the manifold is used between neighbouring points. The distance between these points is adapted according to the curvature of the manifold in the plane. We remark that in the plane of intersection, the 1D unstable manifold may self-intersect due to the projection from an infinite dimensional space. For further details see the companion paper Ref. [5]. Note that it is not possible to compute the infinite-dimensional stable manifold of a DDE.

In this paper, we compute unstable manifolds to study sudden transitions to chaos in a DDE describing a semiconductor laser subject to phase-conjugate feedback (PCF). Physically, understanding such transitions is very important as near such points a small change of parameter, possibly due to noise, could lead to vast changes in the observed dynamics. For example, in the PCF laser this could lead to a switch between stable periodic output and chaotic fluctuations of the laser light. Furthermore, near the bifurcation point, the transients of the system behave chaotically. Specifically, we show here a sudden transition due to the break-up of a torus culminating in a crisis bifurcation and an intermittent transition to chaos.

A laser subject to PCF can be described by the three-dimensional DDE

$$\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\left[-i\alpha G_N(N(t)-N_{\mathrm{sol}}) + \left(G(t)-\frac{1}{\tau_p}\right)\right]E(t) + \kappa E^*(t-\tau) \quad (1)\\
\frac{dN}{dt} &= \frac{I}{q} - \frac{N(t)}{\tau_e} - G(t)\,|E(t)|^2
\end{aligned}$$

for the evolution of the slowly varying complex electric field $E(t) = E_x(t) + iE_y(t)$ and the population inversion $N(t)$. Nonlinear gain is included in the term $G(t) = G_N(N(t)-N_0)(1-\epsilon P(t))$ where $P(t) = |E(t)|^2$ is the intensity. All other parameters were set to realistic values corresponding to a Ga-Al-As semiconductor laser.[2,3] The phase-conjugate feedback term involves the feedback rate κ and the external cavity round-trip time $\tau = 2/3\,\mathrm{ns}$, corresponding to an external cavity length $L_{\mathrm{ext}} \approx 10\,\mathrm{cm}$. In this study, we consider the dimensionless bifurcation parameter $\kappa\tau$. Finally, we note that (1) has $\mathbb{Z}_2$-symmetry given by the transformation $(E,N) \to (-E,N)$. Consequently, every invariant set is either symmetric or has a counterpart under this symmetry. The fact that its solutions are isolated make the PCF laser an ideal test case for numerical continuation[1] and unstable manifold computations.

The general dynamics of the PCF laser consists of stable periodic output interspersed with chaotic dynamics [2]. Furthermore, these periodic solutions

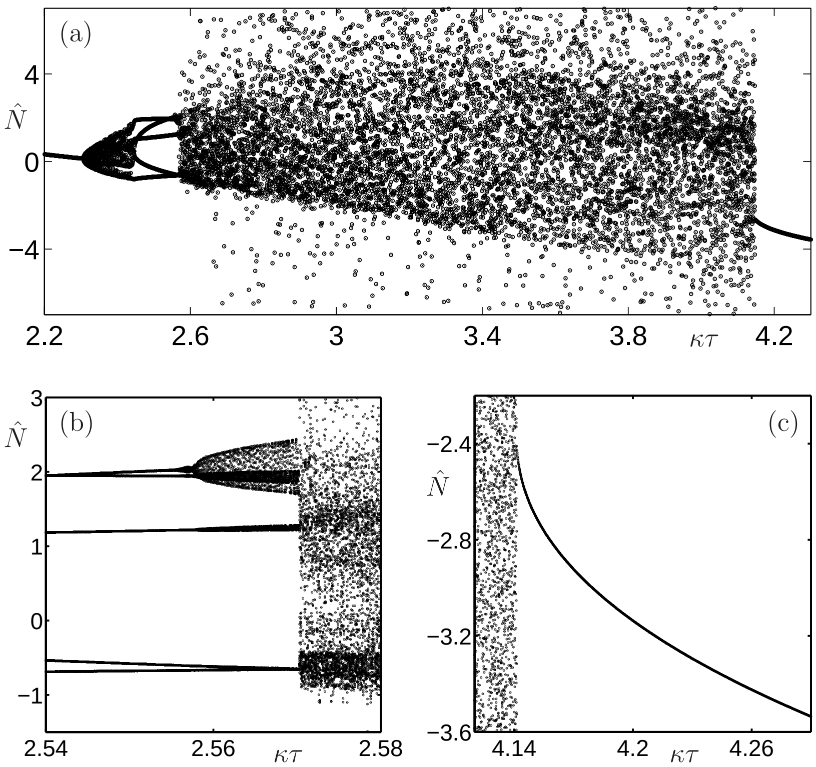

Figure 1. Bifurcation diagrams of the PCF laser obtained by simulation showing the second 'bubble' of complicated dynamics (a), with two enlargements (b) and (c) near the boundaries of the chaotic region

(known as *external cavity modes* of the PCF laser) were shown to be interconnected via an unstable steady state solution. It is the purpose of this paper to study the sudden transitions from the stable periodic operation to the regions of chaos.

Figure 1 shows the second 'bubble' of chaos of the PCF laser. A periodic solution is seen to undergo a torus bifurcation at $\kappa\tau \approx 2.307$. We then observe quasiperiodic modulation until the laser frequency locks to a period five solution, at $\kappa\tau \approx 2.440$, in a saddle-node bifurcation of limit cycles.[3] This periodic solution is itself seen to undergo a torus bifurcation at $\kappa\tau \approx 2.556$, before a sudden jump to the chaotic region is observed at $\kappa\tau \approx 2.570$; see Fig. 1 (b). At $\kappa\tau \approx 4.141$ we observe a second sudden jump in the dynamics of the PCF laser from chaotic dynamics to stable periodic

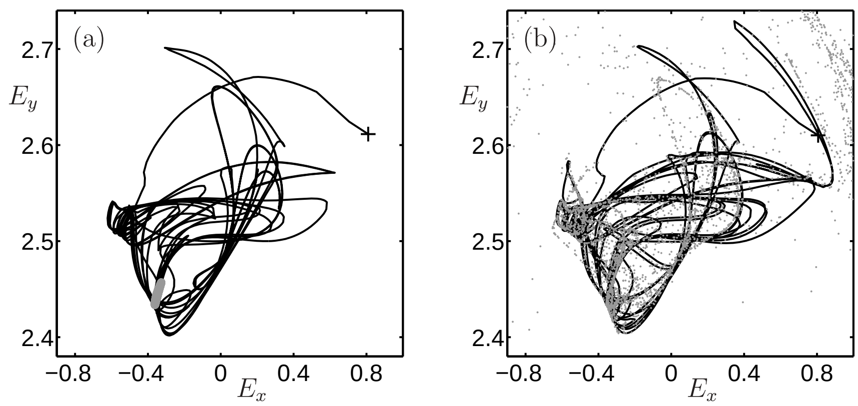

Figure 2. Crisis bifurcation. One branch of the 1D unstable manifold (black curve) of the saddle point (+) and the associated main attractor (grey dots), before the crisis for $\kappa\tau = 2.568$ (a), and after the crisis for $\kappa\tau = 2.572$ (b).

operation; see Fig. 1 (c). This stable periodic solution is created, together with a saddle periodic solution, in a saddle-node bifurcation of limit cycles.[2] In both transitions we see a sudden change in size or shape of the attracting solution.

To shed more light on these transitions, we compute suitable 1D unstable manifolds. Figure 2 (a) shows one branch of the unstable manifold of one of the five saddle points for $\kappa\tau \approx 2.568$, before the bifurcation point, while Fig. 2 (b) shows the same branch of the unstable manifold for $\kappa\tau \approx 2.572$, after the bifurcation. Just prior to the bifurcation, the unstable manifold is confined to the basin of attraction of the torus. Eventually the manifold converges to the attracting invariant circle (in grey) defined by the intersection of the torus with the plane; see Fig. 2 (a). After the bifurcation, the attracting torus has disappeared to be replaced by a chaotic attractor (in grey); see Fig. 2 (b). Notice that the chaotic attractor resembles the unstable manifold prior to the transition and the unstable manifold, which still has essentially the same shape as that shown in Fig. 2 (a), accumulates on this new attractor. This is indicative of a crisis bifurcation.[8]

Figure 3 shows the unstable manifold of a saddle periodic orbit born at $\kappa\tau \approx 4.141$. In each panel a short branch is seen that ends up at an attracting fixed point, while the other branch is longer and more complex. For orientation we overlay the chaotic attractor just before it disappears, for $\kappa\tau = 4.14$. This shows that the long branch of the unstable manifold

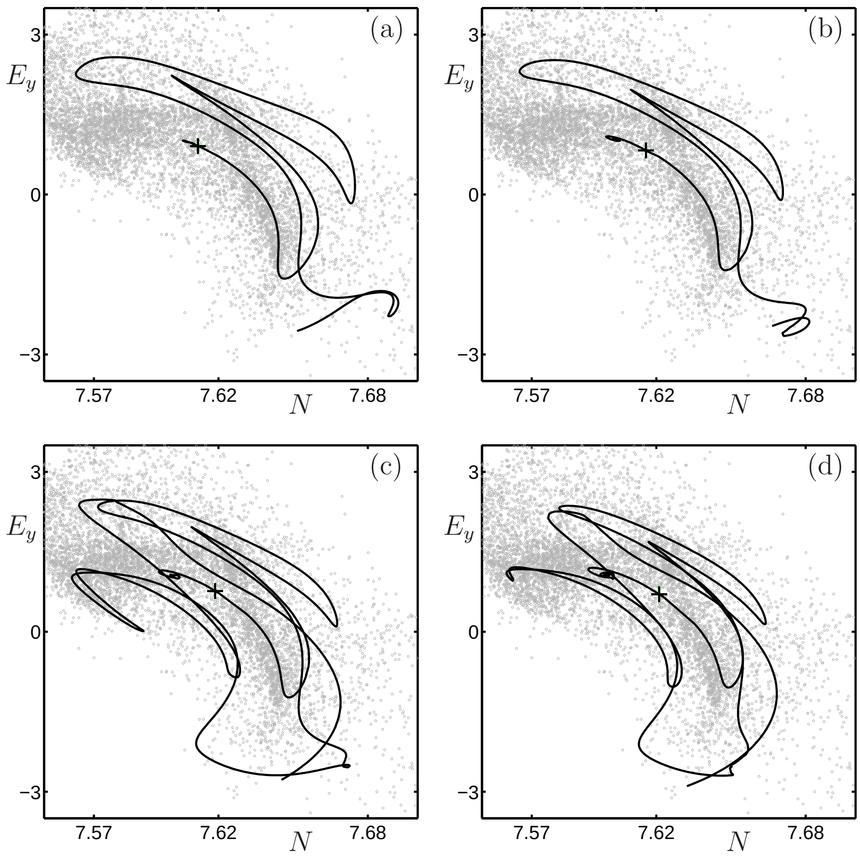

Figure 3. Intermittent transition. Both branches of the 1D unstable manifold of the saddle point (+). The short branch converges to the attractor (×) while the long branch is confined to the chaotic attractor (grey) that exists before the transition. From (a) to (d) $\kappa\tau$ takes the values 4.15, 4.20, 4.25 and 4.30.

essentially resembles the chaotic attractor. As one moves away from the bifurcation point [Figs. 3 (a) to (d)] the short branch grows but always ends up at the attracting fixed point. In contrast, the long branch makes long excursions, 'following' the old chaotic attractor leading to chaotic transients.

This is indicative of an *intermittent transition* (or saddle-node bifurcation taking place on a chaotic attractor). At the transition itself, the long, more chaotic branch (technically the unique unstable branch of the center manifold) forms the chaotic attractor.

In conclusion, we have used a recently developed algorithm for computing 1D unstable manifolds in DDEs to investigate two sudden transitions to chaos in the PCF laser. Specifically, we showed how a route to chaos via the break-up of a torus culminated in a crisis bifurcation. This region of chaos was shown to end abruptly due to a saddle-node bifurcation taking place on a chaotic attractor, also known as an intermittent transition.

More generally, our investigation shows how new numerical tools can be used to provide deeper insight into physical systems modelled by DDEs.

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